# An Experimental Study of Minimum Cost Flow Algorithms

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# Overview

## The Minimum Cost Flow Problem

# 1. The Minimum Cost Flow Problem

# The Minimum Cost Flow Problem

The *minimum cost flow* problem is the following:

- Deliver specified amount of flow from a set of supply nodes to a set of demand nodes in a network.
- There are capacity constraints and costs on the arcs.
- The total cost of the transportation has to be minimized.

# The Minimum Cost Flow Problem

Formal definition:

- Let $G = (V, E)$ be a directed graph.
- We assign for each arc $(i, j) \in E$
  - a lower bound $l_{ij} \geq 0$,
  - an upper bound $u_{ij} \geq l_{ij}$ and
  - a cost $c_{ij}$ (per unit flow).

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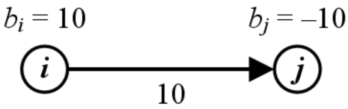
- For each node $i \in V$, we assign a signed supply/demand value $b_i$.



  - If $b_i > 0$, then $i$ is a supply node with $b_i$ supply.
  - If $b_j < 0$, then $j$ is a demand node with $-b_j$ demand.

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    - a cost $c_{ij}$ (per unit flow).
- For each node $i \in V$, we assign a signed supply/demand value $b_i$.
- The goal is to find a feasible flow of minimum total cost.
- The objective function is linear.

R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice-Hall, Inc., 1993.

# The Minimum Cost Flow Problem

This model can be formulated as an LP problem.

## The Minimum Cost Flow Problem

$$\min \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{1}$$

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b_i \qquad \forall i \in V \tag{2}$$

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We usually assume that all input data are integer and we are looking for an integer-valued flow (ILP problem).

# The Minimum Cost Flow Problem

Applications:

- This model can be directly applied in various areas:
    - transportation,
    - logistics,
    - telecommunication,
    - network design,
    - resource planning,
    - scheduling
    - etc.
- It also arises as subproblems of more complex optimization models, such as multicommodity flows.

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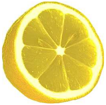
- Provide open source implementations as part of the **LEMON** library.

# 2. Implementation and Testing

# LEMON

Our implementations are part of the **LEMON** combinatorial optimization library.



**LEMON** library:

- **L**ibrary for **E**fficient **M**odeling and **O**ptimization in **N**etworks
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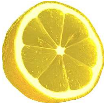

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- It contains highly efficient and well cooperating data structures and algorithms that help solving various optimization tasks related to graphs and networks.
- It is similar to BGL (Boost Graph Library) and LEDA.

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# Test Instances

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- Several benchmark sets of random networks were generated using NETGEN, GRIDGEN and GOTO.
- The largest instances contain millions of nodes and arcs.
- Costs and capacities are in the range $[1..10000]$ and $[1..1000]$, respectively.
- In NETGEN and GRIDGEN instances, there are $\sqrt{n}$ supply and $\sqrt{n}$ demand nodes with total supply $1000\sqrt{n}$.
- The GOTO problems contain single source and single target nodes.

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Generated test instances:

- We usually obtained similar results for the NETGEN and GRIDGEN networks, thus GRIDGEN tests are omitted here.
- GOTO typically generates much harder problems than the other two generators.
- The most important parameter is the density of the graph. The algorithms perform diversely on sparse and dense networks.
- For the sparse graphs, $m \approx 8n$ and for the dense networks, $m \approx \sqrt{n}\,n$.

# Test Instances

Real-world test instances:

- Some real-world problems were also tested.
- They are based on maximum flow instances that arose in medical image processing (`http://vision.csd.uwo.ca/`).
- Random costs are assigned to the arcs and we are looking for a maximum flow of minimum total cost.

# Benchmark System

Benchmark system:

- AMD Opteron Dual Core 2.2 GHz CPU (1 MB cache), 16 GB memory,
- openSUSE 10.1, GCC 4.1.0 compiler, –O3 option.

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4. Network Simplex method
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# 3. Solution Methods

## I. Cycle Canceling Algorithms

# Cycle Canceling Method

**Theorem 1.** Negative cycle optimality condition

A feasible solution $x$ of the minimum cost flow problem is optimal if and only if the residual network $G_x$ contains no directed cycle of negative total cost.

*Note.* The cost of a reversed arc is the opposite of the cost of the original arc: $c_{ji} = -c_{ij}$.

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*Primal method*: it maintains a feasible solution and attempts to reduce the objective function value (the total cost of the flow) at every iteration.

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Implemented algorithms:

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# Cycle Canceling Algorithms

Implemented algorithms:

**SCC**: *Simple Cycle Canceling*

- The Bellman–Ford algorithm is used for finding negative cycles.
- Some practical heuristics were applied to reduce running time.

**MMCC**: *Minimum Mean Cycle Canceling*

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# Cycle Canceling Algorithms

Implemented algorithms:

**SCC**: *Simple Cycle Canceling*

**MMCC**: *Minimum Mean Cycle Canceling*

- It cancels a *minimum mean cycle* at each iteration.
- A simple, well-known strongly polynomial algorithm.
- However, it is extremely slow in practice.

**CAT**: *Cancel and Tighten*

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- It is an improved version of MMCC.
- Actually, it applies a *primal–dual* approach: storing node potentials (the dual solution), it finds negative cycles much faster on average.

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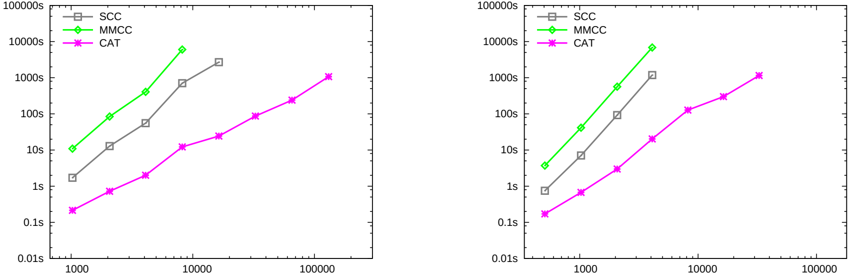
- Actually, it applies a *primal–dual* approach: storing node potentials (the dual solution), it finds negative cycles much faster on average.
- It is also strongly polynomial, but it is much more efficient than the previous two algorithms (both in theory and in practice).

# Cycle Canceling Algorihms

In these charts, the cycle canceling algorithms are compared.
Running times are shown in seconds as a function of the number of nodes (logarithmic scale is used).



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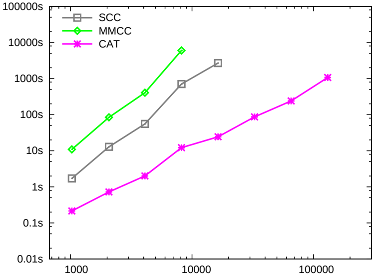

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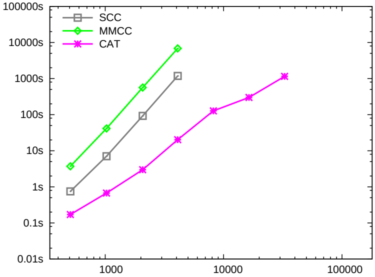

Dense networks (NETGEN)

- **SCC** is 6-8 times faster than **MMCC**.
- **CAT** is an order of magnitude faster than the others.

# Augmenting Path Method

## 3. Solution Methods

## II. Augmenting Path Algorithms

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- At each step, a certain amount of flow is sent from a node with excess to a node with deficit along a shortest path (with respect to the reduced costs).
- If there are no nodes with excess, a primal feasible solution is reached.
- It is also optimal, since the dual feasibility is throughout preserved.

# Augmenting Path Method

The dual solution of the minimum cost flow problem is represented by node potentials.

Another optimality condition (equivalent to Theorem 1).

**Theorem 2.** Reduced cost optimality condition

A feasible solution $x$ of the minimum cost flow problem is optimal if and only if for some node potential function $\pi$, the *reduced cost* of each arc in the residual network $G_x$ is non-negative.

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**Definition.** Reduced cost

For a given set of node potentials $\pi$, the reduced cost of an arc $(i,j)$ is defined as

$$c_{ij}^{\pi} = c_{ij} + \pi(i) - \pi(j).$$

# Augmenting Path Algorithms

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- Simple variant using Dijkstra's algorithm.

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- A faster (polynomial) version of SSP algorithm.
- At each step, we are looking for a shortest path on which at least $\Delta$ amount of flow can be sent.

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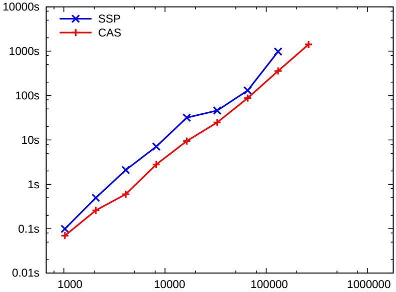
- A faster (polynomial) version of SSP algorithm.
- At each step, we are looking for a shortest path on which at least $\Delta$ amount of flow can be sent.
- If such a path is not found, the value of $\Delta$ is halved and another phase is performed.
- The last phase ($\Delta = 1$) results in a feasible and optimal flow.

# Augmenting Path Algorithms

The augmenting path algorithms are compared in these charts.



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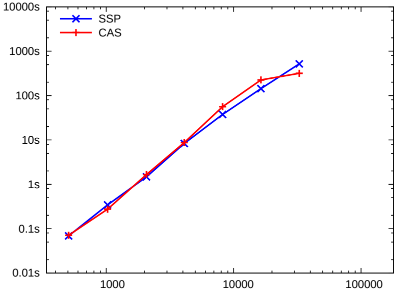

Dense networks (NETGEN)

- **CAS** usually performs better than **SSP**.
- However, if the capacities or the supply/demand values are rather small, then **SSP** is clearly the fastest solution method. (Only a few calls of Dijkstra's algorithm are required.)

# Cost Scaling Method

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Cost scaling method:

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- In the scaling phases, *push* and *relabel* operations are used.

# Cost Scaling Method

Implemented algorithms:

**COS-PR**: *Push-Relabel*

**COS-AR**: *Augment-Relabel*

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# Cost Scaling Method

Implemented algorithms:

**COS-PR**: *Push-Relabel*

- The original variant using local push and relabel operations.

**COS-AR**: *Augment-Relabel*

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# Cost Scaling Method

Implemented algorithms:

**COS-PR**: *Push-Relabel*

**COS-AR**: *Augment-Relabel*

- Instead of the push operations, augmenting paths are found from excess nodes to deficit nodes.
- A path augmentation is equal to several consecutive push operations.

**COS-PAR**: *Partial Augment-Relabel*

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- Goldberg's new idea is applied to this problem: the length of an augmenting path is limited.

Andrew V. Goldberg. *The partial augment-relabel algorithm for the maximum flow problem.* ESA 2008, 466–477, 2008.

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- Goldberg's new idea is applied to this problem: the length of an augmenting path is limited.
- At once, flow is sent on a path consisting of at most $k = 4$ arcs.
- It proved to be a good compromise between the above two methods. It is significantly faster in practice.

Andrew V. Goldberg. *The partial augment-relabel algorithm for the maximum flow problem.* ESA 2008, 466–477, 2008.

# Cost Scaling Heuristics

The performance of the Cost Scaling algorithm highly depends on the applied heuristics.

In our implementations, the following heuristics are used:

- price refinement,
- early termination,
- global update,
- push-look-ahead (only in the push-relabel version).

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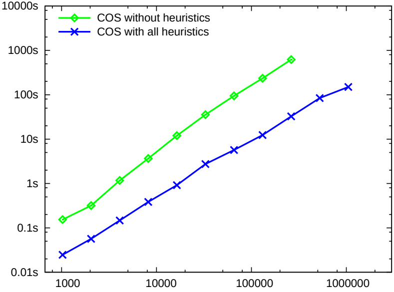

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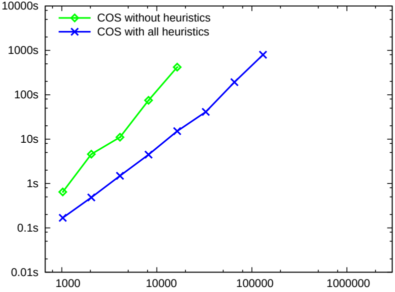

Sparse networks (GOTO)

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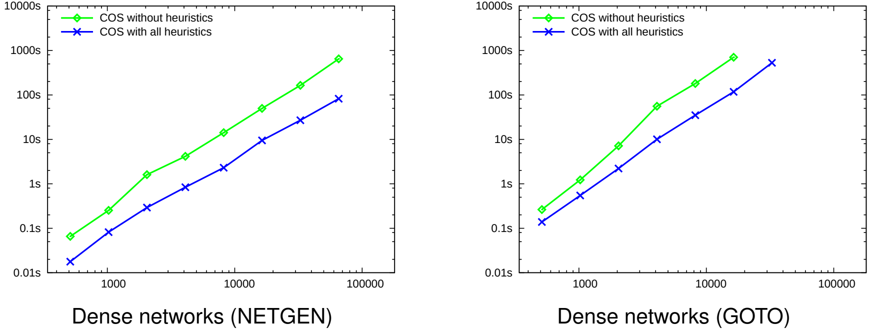

Dense networks (NETGEN)

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- The LP bases are represented by *spanning tree solutions*.
- Such a solution is given by a spanning tree of the network for which the flow values are fixed on all arcs outside the tree (i.e. they have a flow value either at the lower bound or at the upper bound).

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- Such a solution is given by a spanning tree of the network for which the flow values are fixed on all arcs outside the tree (i.e. they have a flow value either at the lower bound or at the upper bound).
- The algorithm maintains a spanning tree with flow values (primal solution) and node potentials (dual solutions).
- At each iteration, we attempt to reduce the objective function value (the total cost of the flow) by moving from one spanning tree solution to another.

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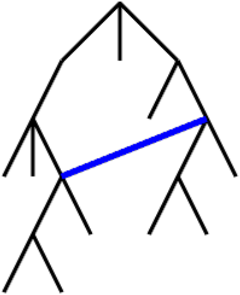
- At each step, a non-tree arc violating the optimality condition is selected.
- This arc is added to the spanning tree (a variable is added to the base). By this operation, a cycle of negative total cost is determined.



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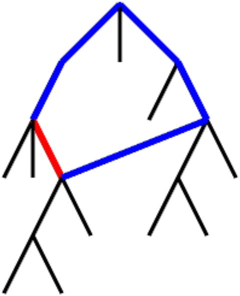
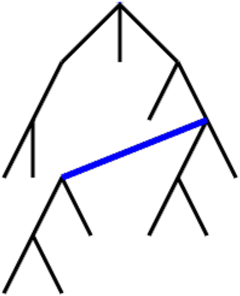
- This arc is added to the spanning tree (a variable is added to the base). By this operation, a cycle of negative total cost is determined.
- This cycle is canceled by augmenting flow along it and one of the saturated (fixed) arcs is removed from the tree. This operation is called *pivot*.



# Network Simplex Method

Primal network simplex algorithm:



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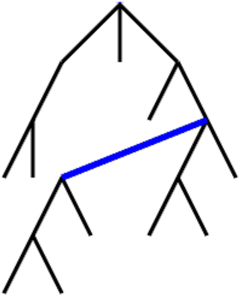

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Actually, this algorithm is a particular variant of the basic primal approach (cycle canceling). Due to the sophisticated method of maintaining spanning tree solutions, a negative cycle can be found much faster (in $O(m)$ time).

# Network Simplex Method

Implementation:

- A complex data structure is required to store and update spanning trees efficiently.
- Several different methods are known for this, e. g. ATI, API, XTI, XPI. One of the most efficient schemes, the XTI method was implemented.

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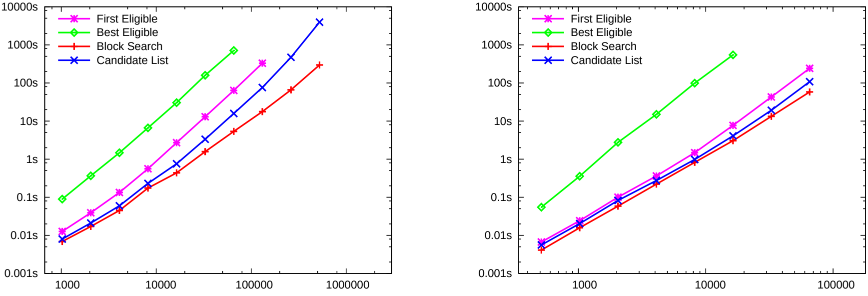
- Apart from maintaining the spanning tree solutions, the most critical operation is the selection of the entering arc (the pivot rule).
- It should be fast, but it should find an arc having reduced cost as small as possible. These requirements are clearly contrary.
- Various pivot rules were implemented applying different approaches. They highly affect the overall running time of the algorithm.

# Network Simplex Pivot Rules



Sparse networks (NETGEN)

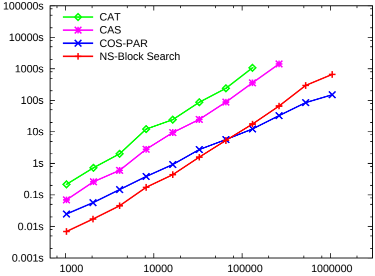
Dense networks (NETGEN)

- **First Eligible** is relatively efficient although it is very simple.
- **Best Eligible** method is by far the slowest one.
- **Block Search** proved to be the most efficient and most robust.
- **Candidate List** is also very efficient.

# 4. Experimental Results

# Comparison

These charts compare our fastest implementations of the four approaches.



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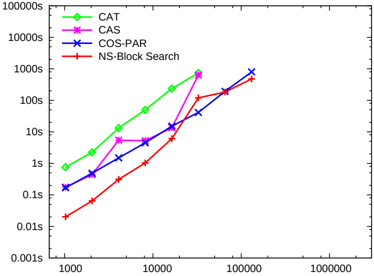

Sparse networks (GOTO)

- Cancel and Tighten (**CAT**) is the slowest among these four implentations.
- Capacity Scaling (**CAS**) is significantly faster.
- The most efficient methods are clearly the Cost Scaling (**COS**) and Network Simplex (**NS**) algorithms.

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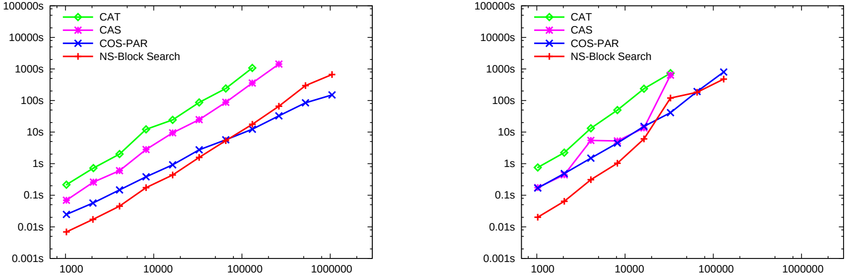

Sparse networks (NETGEN)　　Sparse networks (GOTO)

- **COS** proved to be *asymptotically* faster than all other methods both on sparse and dense networks.
- Therefore, **COS** is the absolute winner on huge networks, especially when they are relatively sparse.
- However, on small and medium sized graphs, **NS** is typically much faster.

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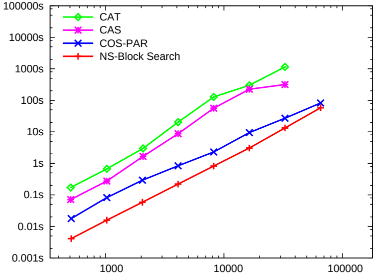

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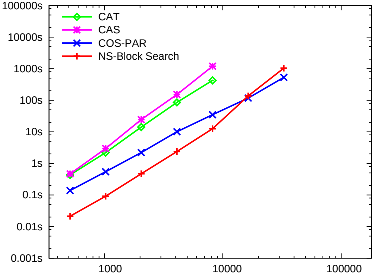

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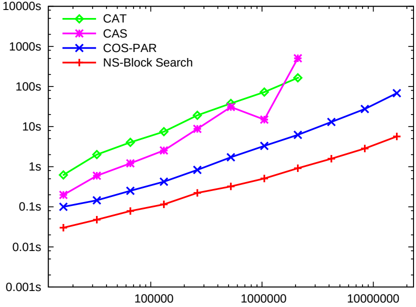
# Comparison

In this chart, the number of nodes is fixed (to 4096) and the running times are shown as a function of the number of arcs.

The largest instance is the full graph containing 16 million arcs.

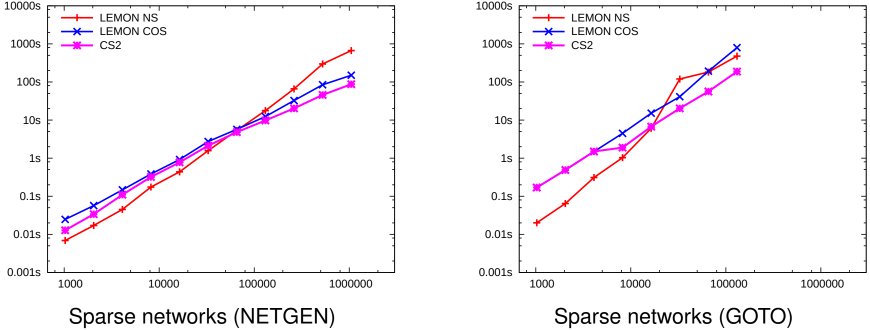


Network Simplex (**NS**) is by far the most efficient in such tests.

# Comparison with Other Solvers

On the following slides, our two fastest implementations, the cost scaling (**COS**) and the network simplex (**NS**) algorithms are compared to widely known efficient solvers.

# Comparison with Other Solvers



Sparse networks (NETGEN)

Sparse networks (GOTO)

- **CS2**: CS2 4.6 (latest version) by A. V. Goldberg (IG Systems).
- It is an efficient implementation of the cost scaling method.
- It proved to be slightly faster than our cost scaling implementation (**COS**).

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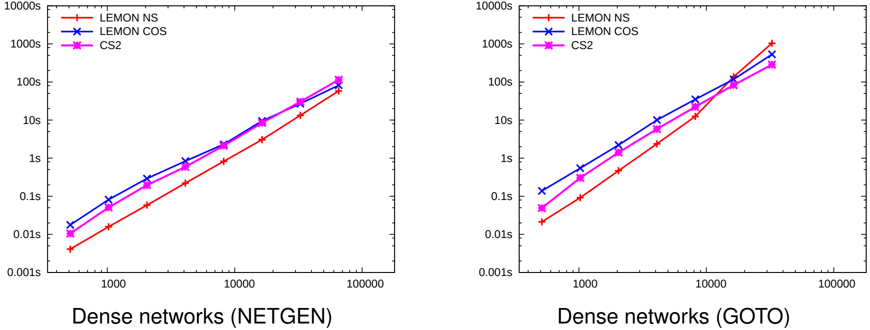

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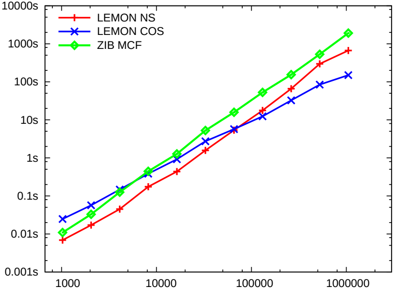

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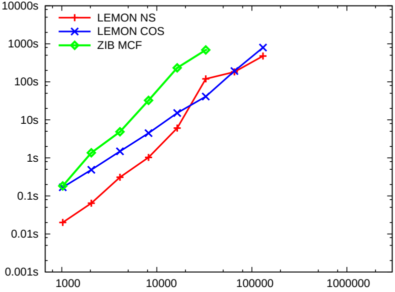

Sparse networks (GOTO)

- **ZIB MCF**: MCF 1.3 (latest version) by A. Lbel (Zuse Institute Berlin).
- It is a network simplex implementation.
- Our **NS** code was much faster on all problem sets.

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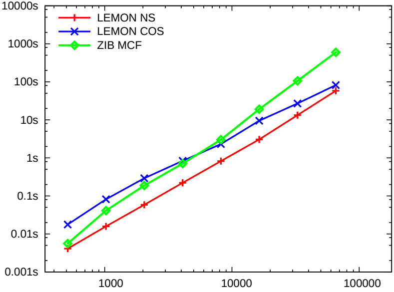

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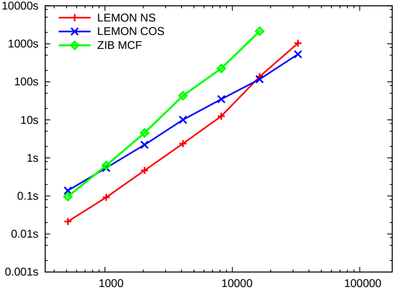

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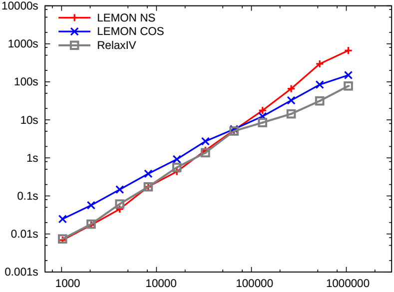

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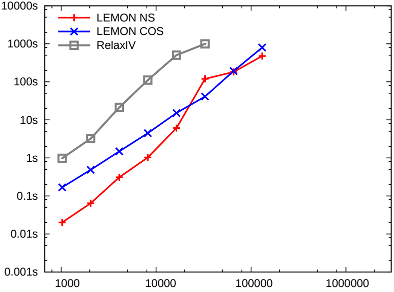

Sparse networks (GOTO)

- **RelaxIV**: an efficient implementation of the relaxation algorithm by D. P. Bertsekas and P. Tseng.
- It proved to be remarkably efficient on NETGEN problems, but it performed extremely poorly on GOTO instances.

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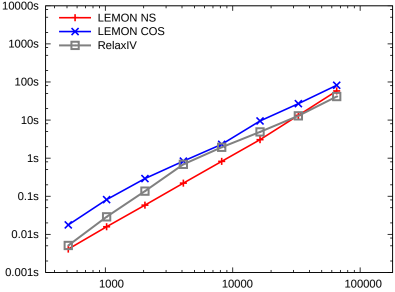

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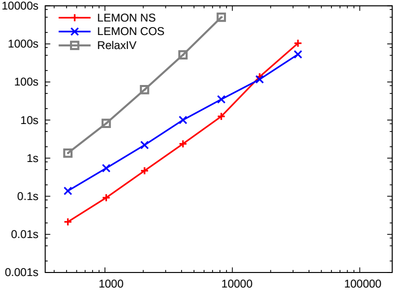

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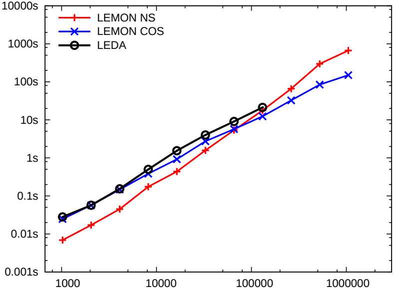

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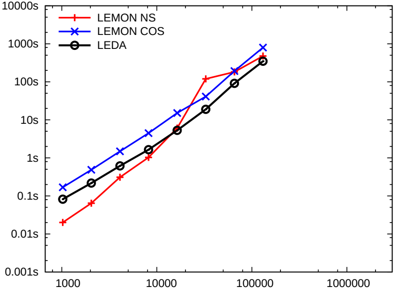

Sparse networks (GOTO)

- **LEDA**: The minimum cost flow method of the LEDA 5.0 C++ optimization library (which is a commercial software).
- LEDA provides an efficient cost scaling implementation.
- Our **COS** code often outperformed it, especially on NETGEN networks.
- Moreover, LEDA failed on the largest instances with *cost overflow* error.

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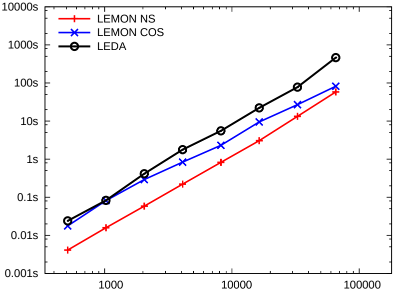

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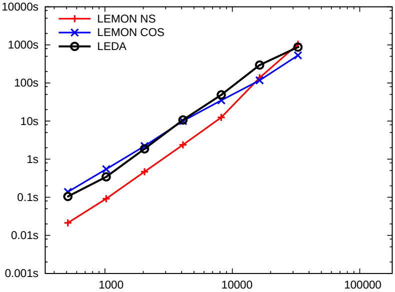

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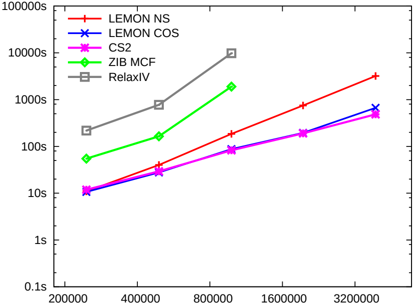
# Comparison on Real-World Networks

This chart show the running times on the real-world networks that arose in segmentation problems in medical image processing.



- Our implementations (**NS** and **COS**) solved these problems efficiently.
- **CS2** was even slightly faster than our **COS** algorithm.
- **ZIB MCF** and **RelaxIV** were not competitive in these tests.

# 5. Summary

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- On small and medium sized graphs and on rather dense graphs, network simplex methods are typically faster.
- Our implementations are available as part of the LEMON open source C++ optimization library.

`http://lemon.cs.elte.hu`